Fidelity induced distance measures for quantum states

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Abstract

Fidelity plays an important role in quantum information theory. In this letter, we introduce new metric of quantum states induced by fidelity, and connect it with the well-known trace metric, Sine metric and Bures metric for the qubit case. The metric character is also presented for the qudit (i.e., d-dimensional system) case. The CPT contractive property and joint convex property of the metric are also studied.

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I. INTRODUCTION

Suppose one has two quantum states ρ and σ, then the fidelity [1, 2] between ρ and σ is given by

$$F(\rho, \sigma) = [\mathrm{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}]^2 \tag{1}$$

Fidelity plays an important role in quantum information theory and quantum computation [3], and it has deep connection with quantum entanglement [4], quantum chaos [5], and quantum phase transitions [6, 7, 8]. However, fidelity by itself is not a metric. It is a measure of the "closeness" of two states. As a metric defined on quantum states, $d(x, y)$ is a function satisfies the following four axioms:

(M1). $d(x, y) \geq 0$ for all states x and y;

(M2). $d(x, y) = 0$ if and only if $x = y$;

(M3). $d(x, y) = d(y, x)$ for all states x and y;

(M4). The triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$ for all states x, y and z.

One may expect that a metric, which is a measure of distance, can be built up from fidelity. Indeed, the following three functions

$$\begin{aligned}
A(\rho, \sigma) &:= \arccos\sqrt{F(\rho, \sigma)}, \\
B(\rho, \sigma) &:= \sqrt{2 - 2\sqrt{F(\rho, \sigma)}}, \\
C(\rho, \sigma) &:= \sqrt{1 - F(\rho, \sigma)},
\end{aligned}$$

exhibit such metric properties. They are now commonly known in the literature as the *Bures angle*, the *Bures metric*, and the *Sine metric* [9, 10, 11], respectively. Based on fidelity, one can generally define a metric $D(\rho, \sigma)$ as $D(\rho, \sigma) := \phi(F(\rho, \sigma))$, where $\phi(t)$ is a monotonically decreasing function of t, and $\phi(F(\rho, \sigma))$ is required to satisfy the axioms M1-M4. From this way, one can define many useful metrics [9, 10, 11]. All of the above three metrics belong to this type and play important roles in quantum information theory.

The purpose of this letter is to introduce a new way to define metric of quantum state based on fidelity. In Sec. II, the new metric is defined. We study the qubit case in detail and naturally connect the new metric with the well-known trace metric, the Sine metric and

Bures metric. In Sec. III, we show that the new metric defined is truly a metric, i.e., it satisfies the axioms M1-M4. The upper bound for the metric is also presented. Conclusion and discussion are made in the last section.

II. METRIC INDUCED BY FIDELITY

Let us define a metric of quantum states as follows:

$$D_T(\rho,\sigma) = \max_{\tau} |F(\rho,\tau) - F(\sigma,\tau)| \tag{2}$$

where the maximization is taken over all quantum states τ (mixed or pure). We call this metric $D_T(\rho,\sigma)$ as the T-metric, and the state τ that attained the maximal is called the optimal state for the metric $D_T(\rho,\sigma)$. The T-metric was first introduced in [12] for pure sates of an abstract transition probability space (T means transition probability), but in this letter, we define it for arbitrary quantum states in the Hilbert space.

The above definition of metric may be not easy to calculate. If τ is a pure state, then fidelity can be simplified as $F(\rho,\tau) = \mathrm{Tr}(\rho\tau)$, hence one can define another version of metric as follows:

$$D_{PT}(\rho,\sigma) = \max_{\tau} |F(\rho,\tau) - F(\sigma,\tau)|, \tag{3}$$

where the maximization is taken over all pure states τ. We call this metric $D_{PT}(\rho,\sigma)$ as the PT-metric, and call the pure state τ that attained the maximal as the optimal pure state.

In this section we consider the case of qubits (two-dimensional quantum system). From the Bloch sphere representation, a qubit is described by a density matrix as

$$\rho(\mathbf{u}) = \frac{1}{2}(\mathbf{I} + \sigma\cdot\mathbf{u})$$

where $\mathbf{I}$ is the 2×2 unit matrix and $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices. Assume $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ are two two states of a qubit, then they can represented by two vectors $\mathbf{u}$ and $\mathbf{v}$ in the Bloch sphere. The Euclidean metric between vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $|\mathbf{u}-\mathbf{v}| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2}$. The trace metric between $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ satisfies $D_{tr}(\rho(\mathbf{u}),\rho(\mathbf{v})) = \frac{1}{2}|\mathbf{u}-\mathbf{v}|$, which is proportional to the Euclidean metric.

For the qubit case, it is well-known that the fidelity has an elegant form([2, 13]):

$$F(\rho(\mathbf{u}), \rho(\mathbf{v})) = \frac{1}{2}[1 + \mathbf{u} \cdot \mathbf{v} + \sqrt{1 - |\mathbf{u}|^2}\sqrt{1 - |\mathbf{v}|^2}]$$

where $\mathbf{u} \cdot \mathbf{v}$ is the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$, and $|\mathbf{u}|$ is the magnitude of $\mathbf{u}$. Then we have the following two Theorems.

Theorem 1. For the qubit case, $D_{PT}(\rho(\mathbf{u}), \rho(\mathbf{v}))$ equals to the trace metric, namely $D_{PT}(\rho(\mathbf{u}), \rho(\mathbf{v})) = \frac{1}{2}|\mathbf{u} - \mathbf{v}| = D_{tr}(\rho(\mathbf{u}), \rho(\mathbf{v}))$.

Proof. Let $\rho = \rho(\mathbf{u})$, $\sigma = \rho(\mathbf{v})$ and $\tau = \rho(\mathbf{w})$, since τ is a pure state, which means $|\mathbf{w}| = 1$, then we get

$$\begin{aligned} |F(\rho, \tau) - F(\sigma, \tau)| &= \frac{1}{2}|(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w}| \\ &\leq \frac{1}{2}|\mathbf{u} - \mathbf{v}| \end{aligned}$$

The optimal pure state is $\tau = \rho(\mathbf{w})$, where $\mathbf{w}$ is a vector that parallels to $\mathbf{u} - \mathbf{v}$. Thus $D_{PT}(\rho(\mathbf{u}), \rho(\mathbf{v})) = \frac{1}{2}|\mathbf{u} - \mathbf{v}| = D_{tr}(\rho(\mathbf{u}), \rho(\mathbf{v}))$.

Theorem 2. For the qubit case, $D_T(\rho(\mathbf{u}), \rho(\mathbf{v}))$ equals to the Sine metric, namely $D_T(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)} = C(\rho, \sigma)$.

Proof. Let $\rho = \rho(\mathbf{u})$, $\sigma = \rho(\mathbf{v})$ and $\tau = \rho(\mathbf{w})$, then one obtains

$$\begin{aligned} &|F(\rho, \tau) - F(\sigma, \tau)| \\ &= \frac{1}{2}\left|(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} + \sqrt{1 - |\mathbf{w}|^2}(\sqrt{1 - |\mathbf{u}|^2} - \sqrt{1 - |\mathbf{v}|^2})\right| \\ &\leq \frac{1}{2}\left[|\mathbf{u} - \mathbf{v}||\mathbf{w}| + \sqrt{1 - |\mathbf{w}|^2}\,|\sqrt{1 - |\mathbf{u}|^2} - \sqrt{1 - |\mathbf{v}|^2}|\right] \\ &\leq \frac{1}{2}\sqrt{|\mathbf{u} - \mathbf{v}|^2 + |\sqrt{1 - |\mathbf{u}|^2} - \sqrt{1 - |\mathbf{v}|^2}|^2} \\ &= \sqrt{1 - F(\rho(\mathbf{u}), \rho(\mathbf{v}))}. \end{aligned} \tag{4}$$

The optimal state is $\tau = \rho(\mathbf{w})$, where $\mathbf{w}$ is a vector that parallels to $\mathbf{u} - \mathbf{v}$, and $|\mathbf{w}| = \frac{|\mathbf{u} - \mathbf{v}|}{\sqrt{1 - F(\rho(\mathbf{u}), \rho(\mathbf{v}))}}$. Thus we have $D_T(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)} = C(\rho, \sigma)$.

This significant result seems surprising, since we know that $\sqrt{1 - F(\rho, \sigma)}$ is the Sine metric introduced in [9, 10, 11], which plays an important role in quantum information processing, but here we can recover it for the qubit case through the definition (2). One

may wonder whether the Bures metric can be obtained by the similar definition. The answer is positive. By using the same approach developed in *Theorem 2*, for the qubit case one can prove that Bures metric $B(\rho,\sigma) = \sqrt{2-2\sqrt{F(\rho,\sigma)}}$ can be expressed in the following equivalent form

$$\begin{aligned} B(\rho,\sigma) \;=\; & \max_{\tau} [|\sqrt{1+|F(\rho,\tau)-F(\sigma,\tau)|} \\ & -\sqrt{1-|F(\rho,\tau)-F(\sigma,\tau)|}|], \end{aligned} \tag{5}$$

where the maximization is taken over all pure and mixed quantum states τ.

III. METRIC CHARACTER OF D_{PT} AND D_T

We come to discuss the case of qudit(i.e., $d \times d$ quantum states). In this case, if τ is a pure state, then the fidelity may have a simple form: $F(\rho,\tau) = \mathrm{Tr}(\rho\tau)$, so we first show the metric character of $D_{PT}(\rho,\sigma)$, where the optimal state τ is restricted to pure state, and then turn to show the metric character of $D_T(\rho,\sigma)$.

We need the following concepts: For two quantum state ρ and σ, let λ_i, $(i = 1, 2, 3, ..., d)$, be all eigenvalues of $\rho - \sigma$, and λ_i's are arranged as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d$. Similarly, let $\lambda_i^{'}$ be all eigenvalues of $\sigma - \rho$. Define $E(\rho,\sigma) := \max \lambda_i$ and define $E(\sigma,\rho) := \max \lambda_i^{'}$, so we know that $\lambda_1 = \max \lambda_i$.

Now we give an interpretation of $E(\rho,\sigma)$. Let ρ and σ be two quantum states, then the following is well known(for example, see [14]):

$$E(\rho,\sigma) = \max_{\tau} \mathrm{Tr}[\tau(\rho-\sigma)], \tag{6}$$

where the maximization is taken over all pure states τ.

Note that in general $E(\rho,\sigma)$ is not a metric, since $E(\rho,\sigma)$ may not equal to $E(\sigma,\rho)$, but we can symmetrize it as:

$$D_S(\rho,\sigma) := \max[E(\rho,\sigma), E(\sigma,\rho)] = \max |\lambda_i|, \tag{7}$$

where $|\lambda_i|$ is the absolute value of λ_i. From the knowledge of matrix analysis, $D_S(\rho,\sigma)$ equals to the spectral metric between ρ and σ, which was defined as the largest singular value of

$\rho - \sigma$, hence we know that $D_S(\rho, \sigma)$ is nothing but the spectral metric. For the qubit case, the $D_S(\rho, \sigma)$ or the spectral metric is also equal to the trace metric, i.e., $D_S(\rho(\mathbf{u}), \rho(\mathbf{v})) = \frac{1}{2}|\mathbf{u} - \mathbf{v}|$. Now we begin to show the metric character of $D_{PT}(\rho, \sigma)$.

Proposition 1. For quantum states ρ and σ, we have $D_{PT}(\rho, \sigma) = D_S(\rho, \sigma)$, i.e, the PT-metric is in fact the same as the spectral metric.

Proof. From the definition $D_{PT}(\rho, \sigma) = \max_{\tau} |F(\rho, \tau) - F(\sigma, \tau)|$, since τ is an arbitrary pure state, we have $F(\rho, \tau) = \mathrm{Tr}(\rho\tau)$. It yields $|F(\rho, \tau) - F(\sigma, \tau)| = |\mathrm{Tr}(\rho\tau) - \mathrm{Tr}(\sigma\tau)|$, then we get $\max_{\tau} |\mathrm{Tr}(\rho\tau) - \mathrm{Tr}(\sigma\tau)| = \max(E(\rho, \sigma), E(\sigma, \rho)) = D_S(\rho, \sigma)$. The Proposition is proved.

Now we know that the PT-metric equals to the spectral metric, so it is a true metric. In the following we shall prove that the T-metric is also a true metric.

Theorem 3. The T-metric $D_T(\rho, \sigma)$ as shown in Eq. (2) is truly a metric, i.e, it satisfies conditions M1-M4.

Proof. From the definition, it is easy to prove conditions M1 and M3 hold. What we need to do is to prove conditions M2 and M4. If $\rho = \sigma$, then of course $D_T(\rho, \sigma) = 0$. If $D_T(\rho, \sigma) = 0$, we will prove $\rho = \sigma$. From the definition, we know that $D_T(\rho, \sigma) \geq D_{PT}(\rho, \sigma)$, so we get $D_{PT}(\rho, \sigma) = 0$, since $D_{PT}(\rho, \sigma)$ is a true metric, we get $\rho = \sigma$. Now we come to prove M4, the triangle inequality $D_T(\rho, \sigma) \leq D_T(\rho, \tau) + D_T(\sigma, \tau)$. $D_T(\rho, \sigma) = \max_{\tau} |F(\rho, \tau) - F(\sigma, \tau)|$, and suppose τ is the state that attains the maximal, so $D_T(\rho, \sigma) = |F(\rho, \tau) - F(\sigma, \tau)|$. We assume that $|F(\rho, \tau) - F(\sigma, \tau)| = F(\rho, \tau) - F(\sigma, \tau)$, then we get $F(\rho, \tau) - F(\sigma, \tau) = F(\rho, \tau) - F(w, \tau) + F(w, \tau) - F(\sigma, \tau) \leq |F(\rho, \tau) - F(w, \tau)| + |F(w, \tau) - F(\sigma, \tau)| \leq D_T(\rho, w) + D_T(w, \sigma)$. Thus one finally has $D_T(\rho, \sigma) \leq D_T(\rho, w) + D_T(\sigma, w)$.

For the qudit case, one does not have the relation $D_T(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ as in *Theorem 2*. However, the numerical computation indicates the following upper bound holds:

$$D_T(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)}. \tag{8}$$

Now we will give the rigorous proof: as was shown in [10], following inequality holds:

$$|F(\rho, \tau) - F(\sigma, \tau)| \leq \sqrt{1 - F(\rho, \sigma)} \tag{9}$$

for arbitrary quantum states ρ, σ, τ. Taking maximal in the left hand of inequality (9), we

get the inequality (8).

IV. DISCUSSION

In summary, we have introduced metric of quantum states induced by fidelity, and connected it with the well-known trace metric, Sine metric and Bures metric for the qubit case. The metric character of $D_T(\rho, \sigma)$ is also presented for the qudit case.

Let us make one more discussion to end the paper.

In quantum information theory, a quantum operation or a quantum channel is representated by a completely positive trace preserving (CPT) map. We say that the metric $d(\rho, \sigma)$ has **the CPT contractive property**, if $d(\phi(\rho), \phi(\sigma)) \leq d(\rho, \sigma)$ always holds, where ϕ is a quantum operation, ρ, σ be two arbitrary density matrices. We wish that a metric is contractive under CPT map (i.e., quantum operation), this has a physical interpretation [9]: a quantum process acting on two quantum states cannot increase their distinguishability.

Also, we wish a metric satisfying the following **joint convex property**: for all $0 \leq \lambda \leq 1$, states $\rho_1, \rho_2, \sigma_1, \sigma_2$, $d(\lambda\rho_1 + (1-\lambda)\rho_2, \lambda\sigma_1 + (1-\lambda)\sigma_2) \leq \lambda d(\rho_1, \sigma_1) + (1-\lambda) d(\rho_2, \sigma_2)$.

The joint convex property also has a physical interpretation [9]: the distinguishability between the states $\lambda\rho_1 + (1-\lambda)\rho_2$ and $\lambda\sigma_1 + (1-\lambda)\sigma_2$, where λ is not known, can never be greater than the average distinguishability when λ is known.

The numerical method ([15]) show that the T-metric

$$D_T(\rho, \sigma) = \max_{\tau} |F(\rho, \tau) - F(\sigma, \tau)|$$

satisfying the CPT contractive property.

How about the joint convex property? We find that, the T-metric D_T is **not** joint convex. However, numerical experiment shows that its square is joint convex, that is, the following holds:

$$D_T^2(\lambda\rho_1 + (1-\lambda)\rho_2, \lambda\sigma_1 + (1-\lambda)\sigma_2) \leq \lambda D_T^2(\rho_1, \sigma_1) + (1-\lambda) D_T^2(\rho_2, \sigma_2)$$

All the above evidences show that metric $D_T(\rho, \sigma)$ is a useful metric in quantum information theory.

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